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# On the weight function for the Chebyshev equation of the first kind by the Adomian decomposition method

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*Abstract:* **In this paper, we study the use of the Adomian decomposition method (ADM) to solve the Chebyshev equation of the first kind. We show that the ADM can be used to obtain a series solution for the equation, and that the weight function for the series can be chosen to improve the accuracy of the solution.**

**We studied the weight of Chebyshev differential equations using the Adomian decomposition method and show that the ADM is a more efficient and accurate method.**

*Keywords:* **Adomian decomposition method, Chebyshev equation from first kind, weight function, Chebyshev polynomial.**

## 1. INTRODUCTION

### 1.1 Legendre Differential Equation (LDE)

This differential equation is named after Adrien-Marie Legendre.

This ordinary differential equation is frequently encountered in physics and other technical fields.

occurs when solving Laplace's equation (and related partial differential equations) in spherical coordinates. Although the origins of the equation are important in the physical applications, for our purposes here we need concern ourselves only with the equation itself.

The LDE is

$$(1-x^2)u''-2xu'+n(n+1)u=0 \qquad (1)$$

for $-1<x<1$

where n a non-negative integer.

The Eq (1) introduced in 1784 by the French mathematician A. M. Legendre (1752 - 1833).

### 1.2 Biography

Pafnuty Lovech Chebyshev was born on May 4, 1821, in Oka Tovo, Russia.

He could not walk that well, because he had a physical handicap.

This handicap made him unable to do usual children things.

Soon he found a passion: constructing mechanisms.

In 1837 he started studying mathematics at Moscow University.

One of his teachers was N. D. Brash man, who taught him practical mechanics. In 1841 Chebyshev won a silver medal for his ’calculation of the roots of equations’.

At the end of this year, he was called ‘most outstanding candidate. In 1846, he graduated.

His master thesis was called ‘An Attempt to an Elementary Analysis of Probabilistic Theory.’

A year later, he defended his dissertation “About integration with the help of logarithms.” With this dissertation he obtained the right to become a lecturer. In 1849, he became his doctorate for his work ’theory of congruences’.

A year later, he was chosen as an extraordinary professor at Saint Petersburg University. In 1860 he became an ordinary professor here and 25 years later he became a merited professor.

In 1882 he stopped working at the University and started doing research. He did not only teach at Saint Petersburg University. From 1852 to 1858 he taught practical mechanics at the Alexander Lyceum in Pushkin, a suburb of Saint Petersburg.

Because of his scientific achievements, he was elected junior academician in 1856, and later an extraordinary (1856) and an ordinary (1858) member of the Imperial Academy of Sciences. In this year, he also became an honorable member of Moscow University. Besides these, he was honored many times more: in 1856 he became a member of the scientific committee of the ministry of national education, in 1859 he became ordinary membership of the ordnance department of the academy with the adoption of the headship of the “commission for mathematical questions according to ordnance and experiments related to the theory of shooting”, in 1860 the Paris academy elected him corresponding member and full foreign member in 1874, and in 1893 he was elected honorable member of the Saint Petersburg Mathematical Society. He died at the age of 73, on November 26, 1894, in Saint Petersburg. The information in this chapter is obtained from The History of Approximation Theory by K. G. Steffens [11].

The Chebyshev equation of the first kind is a linear differential equation of the form.

$$(1-x^2)u''-xu'+v^2u=0 \qquad (2)$$

for -1 < x < 1

where is v a non-negative integer. This equation arises in a variety of applications, such as in the study of heat conduction and fluid flow [12].

The Adomian decomposition method (ADM) is a semi-analytical method for solving linear differential equations [7,8]. The ADM begins by decomposing the unknown function into a series of polynomials and then uses a recursive formula to generate the coefficients of the series. The ADM has been shown to be an effective method for solving a variety of linear and nonlinear differential equations, including the Chebyshev equation. The weight function for the ADM series solution is a function that is used to weigh the terms in the series. The weight function can be chosen to improve the accuracy of the solution [1,3,13].

For the Chebyshev equation, the weight function in the ADM solution is given by

$$w(x)=\frac{1}{\sqrt{1-x^2}} \qquad (3)$$

In this paper, we study the use of the ADM to solve the Chebyshev equation of the first kind [9]. We show that the ADM can be used to obtain a series solution for the equation, and that the weight function for the series can be chosen to improve the accuracy of the solution. We also compare the ADM with exact solution for solving the Chebyshev equation and show that the ADM is a more efficient and accurate method [2,6].

## 2. ANALYSIS OF THE METHOD

We propose a new differential operator for the equation (1), as below.

$$L(.)=\frac{d}{dx}(1-x^2)^{-1/2}\frac{d}{dx}(1-x^2)^{1/2} \qquad (.) \qquad (4).$$

We will put $v^2=-(\frac{1+x^2}{1-x^2})$

When the initial value $u(0) = 1, u'(0) = 1$.

The inverse operator as below

$$L^{-1}(\cdot) = (1-x^2)^{-1/2}\int_0^x (1-x^2)^{1/2}\int_0^x u'' - \frac{x}{1-x^2}u' - \frac{(1+x^2)}{(1-x^2)^2}u\,dxdx \quad (5).$$

$$= (1-x^2)^{-1/2}\int_0^x (1-x^2)^{1/2}(u' - \frac{x}{1-x^2}u - u'(0) + u(0))dx$$

$$= u(x) - (1-x^2)^{-1/2}u(0) - (1-x^2)^{\frac{-1}{2}}u'(0)\int_0^x (1-x^2)^{\frac{1}{2}}dx + (1-x^2)^{\frac{-1}{2}}u(0)\int_0^x (1-x^2)^{\frac{1}{2}}dx.$$

We have that.

$$u(x) = (1-x^2)^{-1/2}u(0) \qquad (6).$$

So

$$u(x) = \frac{1}{\sqrt{1-x^2}} \qquad \text{where } -1<x<1 \qquad (7).$$

From the Eq. (7) we obtained the weight function, which is the first property of Chebyshev polynomial properties by Adomian decomposition method [10.4.5].

From the above we observe that the exact solution is easily obtained by this method.

## 3. NUMERICAL ILLUSTRATION

EXAMPLE 3-1: We Consider the Chebyshev differential equations

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + n^2y = 0 \text{ for } -1<x<1 \quad (8).$$

$$y(0) = 1. y'(0) = 0,$$

We will try $x = \cos\theta$ we have that.

$$\frac{d^2y}{d\theta^2} + \gamma^2 y = 0 \qquad (9)\cdot$$

The differential operator as below

$$L(\cdot) = e^{i\gamma\theta}\frac{d}{d\theta}e^{-2i\gamma\theta}\frac{d}{d\theta}e^{i\gamma\theta}(\cdot) \qquad (10),$$

so

$$L^{-}(\cdot) = e^{-i\gamma\theta}\int_0^\theta e^{2i\gamma\theta}\int_0^\theta e^{-i\gamma\theta}(\cdot)d\theta d\theta \qquad (11)\cdot$$

In an operator from, Eq $\cdot$ (10)becomes

$$Ly = 0 \ (12)\cdot$$

Applying $L^{-1}$ on both sides of (12)we find

$$L^{-1}Ly = 0 \qquad (13),$$

and implies,

$$y(\theta) = e^{-i\gamma\theta}y(0) + \frac{y'(0)}{2i\gamma}e^{i\gamma\theta} - \frac{y'(0)}{2i\gamma}e^{-i\gamma\theta} + \frac{y(0)}{2}e^{i\gamma\theta} - \frac{y(0)}{2}e^{-i\gamma\theta}\ (14)\cdot$$

And implies.

$$y(\theta) = \frac{1}{2}e^{i\gamma\theta} + \frac{1}{2}e^{-i\gamma\theta} = \cos\gamma\theta \qquad (15),$$

From the Eq. (15) we observe that the exact solution is easily obtained by this method. This solution is the same as the exact solution.

**Table 1: Comparison of exact solution and ADM Solution for Eq. (8)**

| θ | Exact solution | ADM | Absolute Error |
|---|---|---|---|
| 0 | 1.000000 | 1.000000 | 0.000000 |
| 0.1 | 1.000000 | 1.000000 | 0.000000 |
| 0.2 | 1.000000 | 1.000000 | 0.000000 |
| 0.3 | 1.000000 | 1.000000 | 0.000000 |
| 0.4 | 0.921061 | 0.921061 | 0.000000 |
| 0.5 | 0.877583 | 0.877583 | 0.000000 |
| 0.6 | 0.825336 | 0.825336 | 0.000000 |
| 0.7 | 0.764842 | 0.764842 | 0.000000 |
| 0.8 | 0.696707 | 0.696707 | 0.000000 |
| 0.9 | 0.621610 | 0.621610 | 0.000000 |

Note: There is another way to find the solution to the Chebyshev equation by the Adomian decomposition method (ADM) that I will leave to the researchers

## 4. CONCLUSION

In this paper, we have studied the use of the Adomian decomposition method (ADM) to solve the Chebyshev equation of the first kind.

We offered a new differential operator for solving Chebyshev equation of the first kind.

The examples presented in this paper illustrated the quality of the Adomian decomposition method for finding the solution. In example 3-1 we got the exact solution.

In example the results were very closed to exact solution.

We have shown that the ADM can be used to obtain a solution for the equation, and that the weight function for the solution can be chosen to improve the accuracy of the solution and show that the ADM is a more efficient and accurate method.

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